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Relaxation heat conduction and generation: an analysis of the semi-infinite body case by method of Laplace transforms

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Abstract—The relaxation equation of heat conduction and generation is solved by method of Laplace transforms for the case of a semi-infinite body and an arbitrary dependence of the surface temperature on time. For the case of equality of the relaxation time of the heat flux (τ_k) and the relaxation time of the internal heat source capacity (τ_g) the Laplace domain solution is inverted analytically, otherwise numerically. Exemplary calculations are carried out for the surface temperature function in the form of a rectangular pulse. The results show that significant differences can occur between the relaxation and parabolic models, in qualitative as well as quantitative terms, which do not disappear for large times. A long-time relaxation solution for $\tau_g = 0$ tends to overlap with the corresponding parabolic solution of a case with heat generation, whilst a long-time relaxation solution for $\tau_g = \infty$ tends to overlap with the corresponding parabolic solution of a case without heat generation.

1. INTRODUCTION

Transport and generation of heat is subject to the phenomenon of relaxation, namely a change in temperature gradient does not cause an instantaneous, corresponding change in heat flux, and, in the case of an internal heat source whose capacity depends on temperature, a change in temperature is not immediately followed by a corresponding change in source capacity. In most engineering applications the relaxation stage of thermal processes can be neglected. However, especially in highly transient heat transfer processes, such as laser pulse heating, the relaxation effects can play an important role.

In general, any relaxation process can be described by a simple exponential model

$$t_r \frac{\partial R_t}{\partial t} + R_t = R_s, \tag{1}$$

where R is the quantity subject to relaxation, and t_r is the relaxation time, representing a finite response time of the system. Subscripts t and s denote the transient and steady state value of R, respectively. For $R_s = R_0 = \text{const}$ and $R_t(0) = 0$, the solution of equation (1) is the function

$$R_t = R_0[1-\exp(-t/t_r)] \tag{2}$$

which is plotted in Fig. 1. Solution (2) and Fig. 1 illustrate the property of the model given by equation (1): R_t approaches R_s over a period of time. A measure of the delay of R_t in relation to R_s is the relaxation time, t_r. As t_r tends to zero R_t tends to R_s.

As a matter of fact, the classical Fourier law in the form

$$\mathbf{q} = -k\nabla T \tag{3}$$

should only be used for modelling steady heat conduction. However, the parabolic equation of heat conduction

$$\frac{\partial T}{\partial t} = a\nabla^2 T + \frac{g}{\rho c_p} \tag{4}$$

which is used to describe both transient and steady heat conduction, was formed by combining equation (3) and the energy conservation equation

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla\cdot\mathbf{q}+g. \tag{5}$$

Because of this, the parabolic equation of heat conduction predicts an infinite speed of heat propagation, i.e. a thermal disturbance at any point in a body is immediately felt at every other point in the body. To bring in a finite speed of heat propagation, Cattaneo [1] suggested the following modification of Fourier's law

$$t_k \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T. \tag{6}$$

Equation (6) can be interpreted as an application of the relaxation model given by equation (1) to heat flux. The elimination of the heat flux vector, $\mathbf{q}$, between equations (5) and (6) results in the well-known hyperbolic equation of heat conduction [1, 2]

NOMENCLATURE

a thermal diffusivity, $k/(\rho c_p)$
c_p specific heat at constant pressure
C_n coefficient of odd-sine series
d arbitrary constant
g capacity of internal heat source (steady)
g_t transient capacity of internal heat source
I_1 modified Bessel function, 1st order
k thermal conductivity
$\mathscr{L}$ Laplace operator
$\mathbf{q}$ heat flux vector
R quantity subject to relaxation
s Laplace variable
t time
t_k relaxation time of heat flux
t_g relaxation time of heat source capacity
t_r relaxation time
T temperature
T_0 reference temperature
w speed of heat propagation, $(a/t_k)^{1/2}$
x, y, z Cartesian coordinates
X, Y, Z dimensionless Cartesian coordinates.

Greek symbols
α constant coefficient
θ dimensionless temperature
θ_i amplitude of thermal pulse
$\theta_s(\tau)$ dimensionless surface temperature
$\bar{\theta}$ transformed dimensionless temperature
ρ density
σ arbitrary constant coefficient
τ dimensionless time
τ_g dimensionless relaxation time of heat source capacity, $t_g/(2t_k)$
τ_i dimensionless duration of thermal pulse
τ_k dimensionless relaxation time of heat flux, 0.5
ψ dimensionless capacity of internal heat source (steady)
ψ_t dimensionless transient capacity of internal heat source.

Superscript
$\bar{}$ transformed variable.

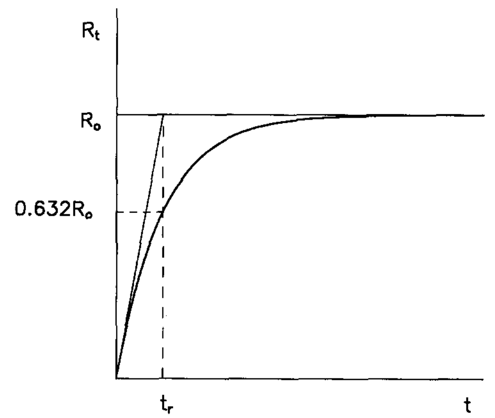

Fig. 1. Transient value R_t of a quantity R subject to relaxation for R_s in the form of step function.

$$t_k \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a\nabla^2 T + \frac{1}{\rho c_p}\left(t_k \frac{\partial g}{\partial t} + g\right) \quad (7)$$

which transmits waves of temperature with a finite speed $w = (a/t_k)^{1/2}$. Though equation (7) takes account of the relaxation of heat flux, it neglects the relaxation of the heat source capacity as in equation (5) g represents the steady state value of the heat source capacity. To eliminate this inconsistency, Malinowski [3] introduced the notion of transient capacity of the heat source. The transient capacity of the source, g_t, is defined by

$$t_g \frac{\partial g_t}{\partial t} + g_t = g. \quad (8)$$

Equation (8) is also based on the relaxation model given by equation (1). It is seen that g_t is related with g in the same way as R_t with R_s. An engineering example of a heat source the capacity of which shows a relaxation behaviour is the ohmic heat source in a conductor carrying an electric current. A change in temperature of the conductor causes a corresponding change in its resistivity, thus a change in the capacity of the source. The source reaches the capacity corresponding to the change in temperature over a period of time.

The set composed of equation (5), in which g is replaced by g_t, equation (6) and equation (8) constitutes the relaxation model of heat conduction and generation. This set can be reduced to the following relaxation equation of heat conduction and generation [3]

$$t_k t_g \frac{\partial^3 T}{\partial t^3} + (t_k + t_g)\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = t_g a \frac{\partial}{\partial t}\nabla^2 T + a\nabla^2 T + \frac{1}{\rho c_p}\left(t_k \frac{\partial g}{\partial t} + g\right). \quad (9)$$

The hyperbolic equation of heat conduction given by equation (7) and the parabolic equation of heat conduction given by equation (4) are special cases of equation (9) for $t_g = 0$ and $t_g = t_k = 0$, respectively.

Equation (9) can be transformed into the following dimensionless form [4]:

$$\tau_g \frac{\partial^3 \theta}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \theta}{\partial \tau^2} + 2\frac{\partial \theta}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\nabla^2 \theta + \nabla^2 \theta + 2\frac{\partial \psi}{\partial \tau} + 4\psi, \quad (10)$$

where $\theta = T/T_0$ is dimensionless temperature, $\tau = t/(2t_k)$ is dimensionless time, $\tau_g = t_g/(2t_k)$ is the dimensionless relaxation time of the heat source capacity, $X = wx/(2a)$, $Y = wy/(2a)$, $Z = wz/(2a)$ are the dimensionless Cartesian coordinates, and $\psi = gt_k/(\rho c_p T_0)$ is the dimensionless (steady) capacity of the internal heat source.

The parabolic, hyperbolic and relaxation equations of heat conduction given by equations (4), (7), (9) and (10), respectively, have been derived on the assumption that the thermophysical properties are constants.

Several special cases of equation (10) have been studied analytically and numerically by Malinowski [3–7]. In ref. [3] he solved analytically a one-dimensional case for a step change in temperature at the surface, the ratio of relaxation times (t_g/t_k) equal to unity, and a linear dependence of the heat source capacity on temperature $(\psi \sim \theta)$. In ref. [4] he studied analytically a number of zero-dimensional cases for various expressions for heat source capacity. He also discussed the physical sense of the relaxation model there. In ref. [5] he presented an analytical solution for a one-dimensional case, in which the relaxation of heat flux is neglected, for a step change in temperature at the surface, various values of t_g/t_k, and $\psi \sim \theta$. In ref. [6] he analysed numerically the temperature field in the semi-infinite body due to a step change in heat flux at the surface for various values of t_g/t_k, and $\psi \sim \theta$. In ref. [7] he studied numerically the evolution of normal zones in a composite superconductor after instantaneous dissipation of a finite amount of energy in the conductor. The calculations were performed for a one-dimensional model, a non-linear dependence of the heat source capacity on temperature, and various values of t_g/t_k.

In this paper we solve the relaxation equation of heat conduction and generation by method of Laplace transforms for the case of a semi-infinite body, an arbitrary dependence of the surface temperature on time, and a linear dependence of the heat source capacity on temperature. For $t_g/t_k = 1$ the solution in the Laplace domain is inverted analytically, otherwise numerically.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

For a linear dependence of heat source capacity on temperature given by the following:

$$\psi = \alpha\theta \quad (11)$$

a one dimensional case of equation (10) is

$$\tau_g \frac{\partial^3 \theta}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \theta}{\partial \tau^2} + 2(1-\alpha)\frac{\partial \theta}{\partial \tau} = \tau_g \frac{\partial^3 \theta}{\partial \tau\, \partial X^2} + \frac{\partial^2 \theta}{\partial X^2} + 4\alpha\theta. \quad (12)$$

The initial conditions for the present problem are

$$\theta(X, 0) = 0 \quad (13a)$$

$$\frac{\partial \theta}{\partial \tau}(X, 0) = 0 \quad (13b)$$

$$\frac{\partial^2 \theta}{\partial \tau^2}(X, 0) = 0. \quad (13c)$$

The boundary conditions are

$$\theta(0, \tau) = \theta_s(\tau) \quad (14a)$$

$$\theta(\infty, \tau) = 0. \quad (14b)$$

3. ANALYTICAL SOLUTION

3.1. *Relaxation model*

The Laplace transform of equation (12) including the initial conditions given by equations (13) is

$$\frac{d^2\bar{\theta}(X,s)}{dX^2} - b\bar{\theta}(X,s) = 0, \quad (15)$$

where

$$\bar{\theta}(X,s) = \mathscr{L}[\theta(X,\tau)] \quad (16a)$$

$$b = s^2 + 2s - 2\alpha(s+2)/(\tau_g s + 1). \quad (16b)$$

The transformed boundary conditions (14) are

$$\bar{\theta}(0,s) = \bar{\theta}_s(s) \quad (17a)$$

$$\bar{\theta}(\infty,s) = 0. \quad (17b)$$

The solution of equation (15) satisfying boundary conditions (17) is the function

$$\bar{\theta}(X,s) = \bar{\theta}_s(s)\bar{f}_1(X,s), \quad (18)$$

where

$$\bar{f}_1(X,s) = \exp(-X\sqrt{b}). \quad (19)$$

For $\tau_g = 0.5$, or $t_k = t_g$, equation (16b) reduces to

$$b = s^2 + 2s - 4\alpha. \quad (20)$$

For b given by equation (20) we can invert solution (18) analytically. First, we determine $f_1(X, \tau)$ with the help of the Laplace-transform tables included in ref. [8]. The procedure employed is the same as that used and described in ref. [3]. Next, we make use of the convolution theorem. After some manipulations we arrive at:

for $\tau < X$

$$\theta(X, \tau) = 0 \tag{21a}$$

for $\tau > X$

$$\theta(X,\tau) = \exp(-X)\theta_s(\tau - X) + X(1+4\alpha)^{1/2} \times \int_X^\tau \theta_s(\tau-\varepsilon)\exp(-\varepsilon)I_1\{[(1+4\alpha)(\varepsilon^2 - X^2)]^{1/2}\} \times (\varepsilon^2 - X^2)^{-1/2}\,d\varepsilon. \tag{21b}$$

3.2. *Parabolic model*

For the parabolic case equation (12) reduces to

$$2\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial X^2} + 4\alpha\theta. \tag{22}$$

The Laplace transformation of equation (22) with regard to the initial condition given by equation (13a) has the same form of equation (15) as the transformation of the relaxation equation (12). But in this case the coefficient b is expressed as

$$b = 2(s - 2\alpha). \tag{23}$$

For the parabolic model, the solution of equation (15) with the boundary conditions (17) also has the form of equation (18). Making use of the Laplace-transform tables [8] and the convolution theorem, we obtain for $X > 0$ the following time domain solution to the parabolic boundary value problem given by equations (22), (13a) and (14)

$$\theta(X,\tau) = X\int_0^\tau \theta_s(\tau-\varepsilon)\exp[2\alpha\varepsilon - X^2/(2\varepsilon)] \times (2\pi\varepsilon^3)^{-1/2}\,d\varepsilon. \tag{24}$$

4. SEMI-ANALYTICAL SOLUTION OF RELAXATION MODEL

We invert numerically the solution in the Laplace domain given by equation (18) for b given by equation (16b) and for $\theta_s(\tau)$ defined as follows:

$$\theta_s(\tau) = \begin{cases} \theta_i & \text{for} \quad 0 \leqslant \tau < \tau_i \\ 0 & \text{for} \quad \tau > \tau_i \end{cases}. \tag{25}$$

The Laplace-transformed equation (25) is

$$\bar{\theta}_s(s) = \frac{\theta_i}{s}[1 - \exp(-\tau_i s)]. \tag{26}$$

We use the method proposed by Papoulis [9]. This method depends on introducing the new independent variable β, defined by $\cos\beta = \exp(-\sigma\tau)$, into the original function, followed by an expansion of the original function into an odd-sine series. According to this method the numerical evaluation of $\theta(X, \tau)$ can be determined from the following partial series:

$$\theta(X,\tau) \cong \exp(-d\tau)\sum_{n=0}^{N} C_n \times \sin\{(2n+1)\arccos[\exp(-\sigma\tau)]\}, \tag{27}$$

where

$$C_n = \frac{4^{n+1}}{\pi}\sigma\bar{\theta}(X, s_n) - \sum_{i=0}^{n-1}\frac{2i+1}{2n+1}\frac{(2n+1)!}{(n+i+1)!\,(n-i)!}C_i \tag{28}$$

$$s_n = d + (2n+1)\sigma. \tag{29}$$

Original function $\theta(X, \tau)$ is determined in terms of the values of the transformed function $\bar{\theta}(X, s)$ on a sequence of $N+1$ equidistant points s_n given by equation (29). As N tends to infinity, the right hand side of equation (27) tends to $\theta(X, \tau)$. d and σ are arbitrary constants. The value of σ depends on the time interval in which $\theta(X, \tau)$ has to be best described. Since for $\tau \to 0$ $s \to \infty$, and for $\tau \to \infty$ $s \to 0$, for small values of τ a large value of σ should be chosen, whereas for large values of τ the value of σ should be small. The constant d must be sufficiently large so that the transformed function $\bar{\theta}(X, s)$ should exist for points s_n.

5. RESULTS AND DISCUSSION

Using solutions (21), (24) and (27), for $\theta_s(\tau)$ given by equation (25), we calculated a number of values of $\theta(X, \tau)$ in order to examine temperature distributions in the body. Solutions of the parabolic case are employed for the purpose of comparison as well as to verify the numerical method of inversion of Laplace transforms. The computations were performed in extended precision arithmetics with 19–20 significant digits, which enabled us to employ 25 terms in the partial sum given by equation (27). The results of calculations are presented in Figs. 2–6. In Figs. 2, 3 and 6 the relaxation solutions are accompanied by two parabolic solutions. The upper dashed line is the parabolic solution corresponding to the relaxation solutions. The lower dashed line represents the parabolic solution for $\psi = 0$. These two parabolic solutions can be considered to be limiting cases for the long-time relaxation solutions for $\tau_g = 0$ and $\tau_g = \infty$, respectively (see Fig. 6). The reasons for this are as follows. For large times the wave character of relaxation solutions decays. Furthermore, when $\tau_g \to 0$ the effect of the heat source inertia on the temperature field becomes negligible, and when $\tau_g \to \infty$ the transient capacity of the source, ψ_t, tends to zero.

Figure 2 gives the profiles of dimensionless temperature in the body at three dimensionless times of $\tau = 2$, 4 and 8. By comparison, the dimensionless relaxation time of heat flux $\tau_k = 0.5$. It is seen that the relaxation solutions are of the wave nature. The energy of the thermal pulse, at first concentrated at the wavefront, dissipates in the body as the front moves. As the time passes the wave character of relaxation solutions vanishes. For $\tau = 8$ (Fig. 2c) the shape

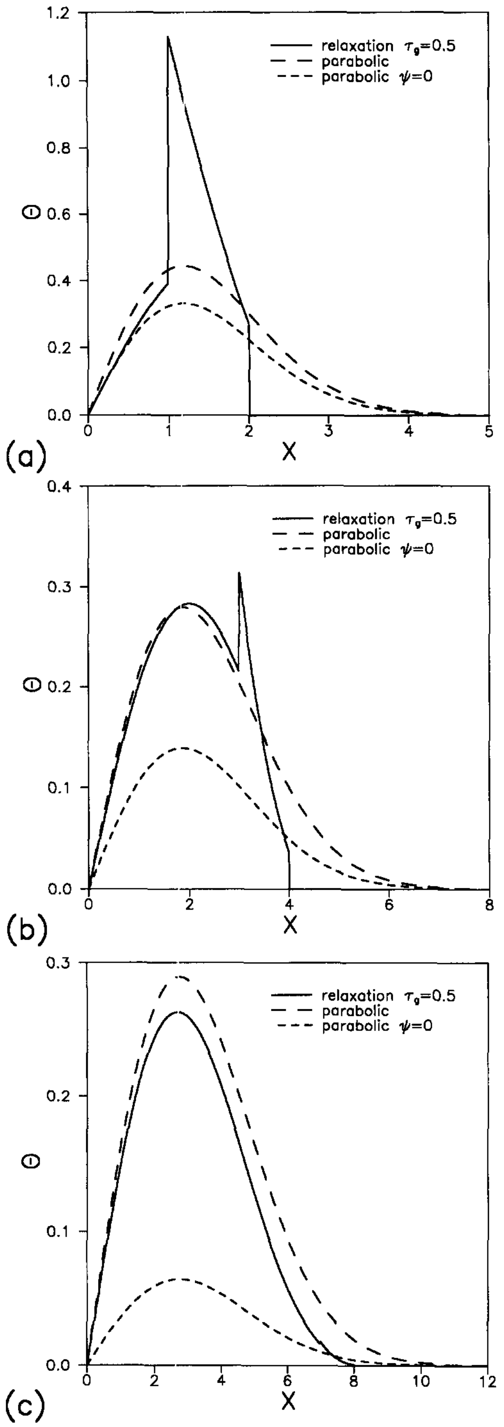

Fig. 2. Dimensionless temperature distribution obtained with analytical solutions for three times: (a) $\tau = 2$; (b) $\tau = 4$; (c) $\tau = 8$. $\theta_i = 2$, $\tau_i = 1$, $\alpha = 0.1$.

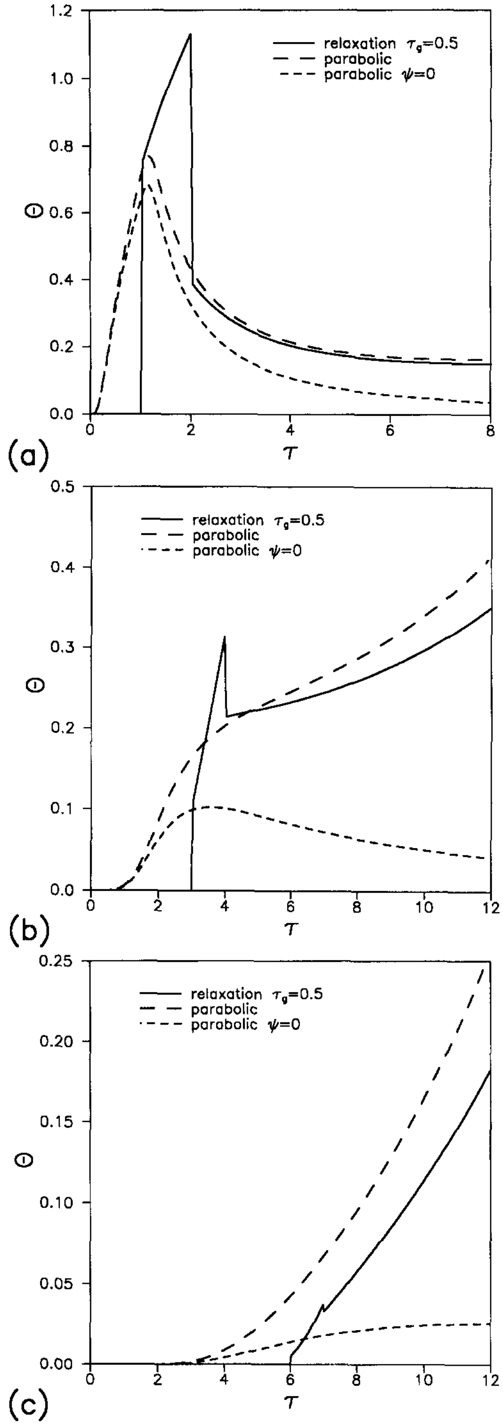

Fig. 3. Dependence of dimensionless temperature on dimensionless time obtained with analytical solutions for three dimensionless coordinates: (a) $X = 1$, (b) $X = 3$, (c) $X = 6$. $\theta_i = 2$, $\tau_i = 1$, $\alpha = 0.1$.

of the relaxation temperature profile does not vary a lot from the shape of the parabolic profiles.

Shown in Fig. 3 are the dependencies of the dimensionless temperature on the dimensionless time at three points on the axis of $X = 1$, 3 and 6. In the case of the parabolic solutions we observe an instantaneous rise in temperature at each of these points after a temperature pulse has been imposed at the boundary, whereas in the case of the relaxation solution, at each of these points the temperature jumps from zero to a finite value after a delay resulting from a finite velocity of the heat wave. The wavefront reaches point $x = x_f$ after time $t_f = x_f/w$, thus $X_f = \tau_f$.

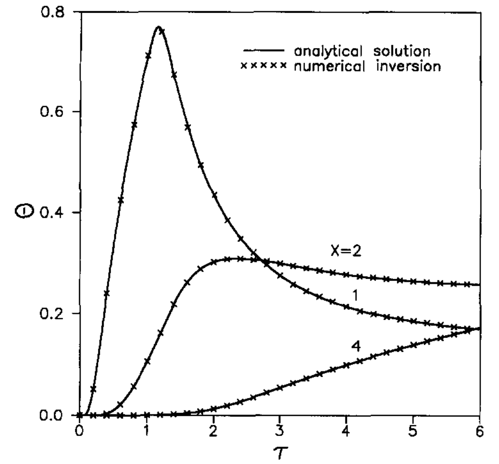

Fig. 4. Comparison of analytical solutions and solutions achieved by the numerical Laplace inversion for the parabolic case. $\theta_i = 2$, $\tau_i = 1$, $\alpha = 0.1$, $d = 0.15$, $\sigma = 0.2$.

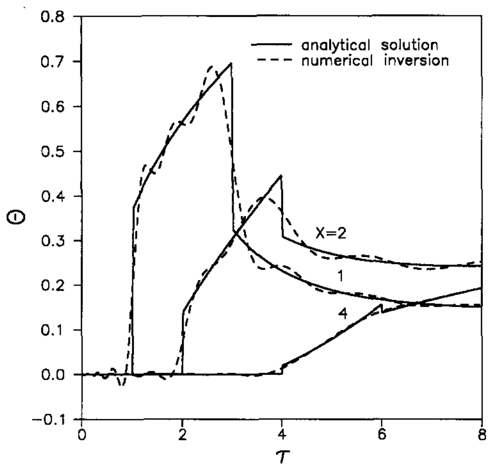

Fig. 5. Comparison of analytical solutions and solutions achieved by the numerical Laplace inversion for the relaxation case. $\theta_i = 2$, $\tau_i = 2$, $\tau_g = 0.5$, $\alpha = 0.1$, $d = 0.15$, $\sigma = 0.2$.

Figures 4 and 5 display the comparison of the analytical results with the results achieved by the numerical inversion of Laplace transforms. The numerical calculations were performed for $N = 24$, $d = 0.15$ and $\sigma = 0.2$. For the parabolic model (Fig. 4) the agreement is very good. In the case of the relaxation model (Fig. 5) some oscillations of the numerical solutions are seen. These oscillations are smaller for points more distant from the surface of the body (see also Fig. 6). However, considering the wave nature of the relaxation solutions, the approximations obtained can be estimated as good.

Figure 6 shows the dimensionless temperature at point $X = 6$ as a function of dimensionless time for $\tau_g = 0.01$, 0.5, 2 and 100. For $\tau_g = 0.5$ there are two solutions: analytical given by a solid line and numerical represented by crosses. For other values of τ_g the solutions are numerical. It is seen that for large times the relaxation solution for $\tau_g = 0.01$ lies in the proximity of the parabolic solution of a case with heat

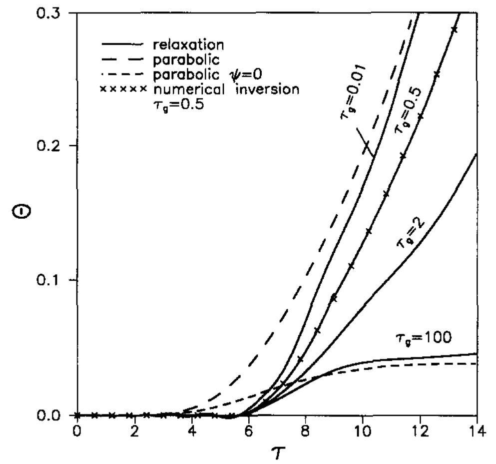

Fig. 6. Dependence of dimensionless temperature on dimensionless time for $X = 6$ and for four dimensionless times of heat source capacity. The curve $\tau_g = 0.5$ is an analytical solution, while the curves $\tau_g = 0.01$, 2 and 100 are solutions obtained with the numerical Laplace inversion. The crosses represent a numerical solution for $\tau_g = 0.5$. $\theta_i = 1$, $\tau_i = 3$, $\alpha = 0.1$, $d = 0.15$, $\sigma = 0.15$.

generation, while the relaxation solution for $\tau_g = 100$ lies in the proximity of the parabolic solution of a case without heat generation. The differences between the relaxation solutions and the related parabolic solution (upper dashed line) do not decrease for large times, which results from the cumulation of the effect of the heat source inertia.

6. CONCLUDING REMARKS

The hyperbolic equation of heat conduction is based on a simple, exponential model of the heat flux relaxation. The same exponential model can be employed to describe the relaxation of the internal heat source capacity. As a result we obtain a more general relaxation equation of heat conduction and generation, which for special cases reduces to the hyperbolic or parabolic equation of heat conduction.

A characteristic feature of many hyperbolic solutions is that they tend to approach the corresponding parabolic solutions at the limit. In the case of relaxation solutions the differences between them and the related parabolic solutions do not vanish as time rises to infinity. This results from the cumulation of the effect of the internal heat source inertia.

Each relaxation solution is related to two characteristic parabolic solutions. The first of them is the solution for the heat source capacity equal to zero, the second one is the solution for the heat source capacity equal to its steady state value. These two parabolic solutions border the region in which long-time relaxation solutions, for various values of the dimensionless relaxation time of heat source capacity (τ_g), are situated. When τ_g changes from zero to infinity the relaxation solution moves from the parabolic solution for a case with heat generation towards the parabolic solution for a case without heat generation. This results directly from the features of the relaxation model of the source, as for $\tau_g = 0$ the source has no inertia, and for $\tau_g = \infty$ the transient capacity of the source equals zero.

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